

Extracting the electro-magnetic pion form factor from QCD in a finite volume

Hidegori Fukaya^{1,*} and Takashi Suzuki^{1,†}

¹*Department of Physics, Osaka University, Toyonaka, Osaka 560-0043 Japan*

Abstract

We consider finite volume effects on the electromagnetic form factor of the pion. We compute the pseudoscalar-vector-pseudoscalar correlator in the ϵ expansion of chiral perturbation theory up to the next-to-leading order and find a way to remove the dominant part, which comes from a contribution of the pion zero-mode. Inserting non-zero momentum to relevant operators (or taking a subtraction of the correlators at different time-slices), and taking an appropriate ratio of them, one can automatically cancel the zero-mode's contribution, which becomes non-perturbatively large $\sim \mathcal{O}(100\%)$ in the ϵ regime. The remaining finite volume dependence, which comes from the non-zero momentum modes, is shown to be perturbatively small even in such an extremal case. Since the zero-mode's dominance is universal in any finite volume scaling, and we do not rely on any particular feature of the ϵ expansion, our method has a wide application to many other correlators of QCD.

*Email:hukaya@het.phys.sci.osaka-u.ac.jp

†Email:suzuki@het.phys.sci.osaka-u.ac.jp

I. INTRODUCTION

The electro-magnetic form factor of the charged pions is one of the fundamental low-energy quantities in Quantum Chromodynamics (QCD). Experimentally, it is related to the pion charge radius $\langle r^2 \rangle_V$ through the relation

$$\langle r^2 \rangle_V = 6 \frac{dF_V(q^2)}{dq^2} \Big|_{q^2=0}, \quad (1)$$

where $F_V(q^2)$ denotes the electro-magnetic form factor at the momentum transfer q^2 . In terms of chiral perturbation theory (ChPT), it is related to the low-energy constant L_9 (or l_6 in the $SU(2)$ case), which appears at the next-to-leading order (NLO) in the chiral Lagrangian [1, 2].

However, it is still a non-trivial task for lattice QCD to fully reproduce or understand the low-energy behavior of the pion form factors. In fact, the lattice data of the pion charge radius have been sizably lower than the experimental value $\langle r^2 \rangle_V = 0.452(11) \text{ fm}^2$ (see the recent review in [3]). It is only recently that consistent values of $\langle r^2 \rangle_V$ were reported by simulations near the physical point [4–6]. According to ChPT, it is known that the pion charge radius shows a logarithmic divergence as the pion mass goes to zero. Thus, we may recognize that our simulated pion masses are too large to reproduce the logarithmic divergence, unless we directly simulate QCD near the chiral limit. Namely, in order to examine the chiral logarithm of the pion charge radius, it is essential to simulate lattice QCD in the very vicinity of the chiral limit.

Although current computational resources allow us to simulate QCD near the physical point, one should carefully take two sources of systematic effects into account in such simulations. One is the cut-off effects, especially those come from breaking of the chiral symmetry. When the simulated quark mass is as small as the typical breaking scale of the chiral (flavor) symmetry (it is typically $\sim \Lambda_{\text{QCD}}^3 a^2$ for the improved Wilson or staggered fermions, where Λ_{QCD} is the QCD scale and a denotes the lattice spacing), it is known that the chiral logarithm is largely distorted. The low-lying Dirac eigenvalue spectrum, for example, is a quantity sensitive to such discretization effects [7].

Another source which may change the chiral behavior is the finite size of the lattice volume. In the literature, it is often mentioned that the lattice size L should satisfy $m_\pi L > 4$, where m_π is a simulated pion mass [8], to suppress the finite size effect at a few % level.

Since the computational cost for inverting the Dirac operator increases as m_π decreases, it is demanding to keep $m_\pi L$ to be large enough. Especially when we want to keep a good chiral symmetry to avoid the former discretization effects on the chiral logarithm, and use a fermion formulation such as overlap or domain-wall fermions, the available range of $m_\pi L$ is quite limited.

This naive criterion about $m_\pi L$, however, comes from the fact that the zero-momentum mode of pions can propagate wrapping around the lattice volume, whose contribution is typically given by $\exp(-m_\pi L)$. For the excited pion states, the finite volume effects are much smaller, since their discrete energy satisfies $E_\pi > 2\pi/L$ in a finite volume, and $E_\pi L > 2\pi$. Therefore, if we can eliminate or reduce the dominant contribution from the pion's zero-momentum mode, one should be able to extract the low-energy quantities even on a small lattice.

In this work, we consider the “worst” case, the so-called ϵ regime of QCD, to show that the above strategy actually works even in such an extremal situation. In the ϵ regime, $m_\pi L < 1$, and the finite volume effects are generally $\sim 100\%$ and we receive a non-perturbatively large correction from the pion zero-mode. However, using the ϵ expansion of ChPT [9], we compute the pseudoscalar-vector-pseudoscalar three-point function, and find a way to automatically cancel the dominant part of them. Since the zero-mode contribution has no space-time dependence, two simple steps are enough to achieve this:

1. inserting non-zero momenta to relevant operators (or taking a subtraction of the correlators at different source points when one or two of the inserted momenta are zero).
2. taking ratios of them.

We also compute the NLO corrections and show that these effects are actually suppressed by $1/F^2 L^2$, where F denotes the pion decay constant. The preliminary result of this work has already appeared in Ref. [10], and applied to numerical works by JLQCD collaboration [5, 6].

Here, we would like to remark the difference of our new approach from the conventional ones in the ϵ regime. In the previous works, the ϵ expansion was used to disentangle the low-energy constants [11–17], using a bunch of Bessel functions, from the lattice data which were largely contaminated by the finite volume effects. In this work, we use (the ϵ expansion of) ChPT in more indirect way : just for finding the combination of the correlators which

has a small sensitivity to the volume. As we will see in the following sections, this idea makes the analysis in the ϵ regime of QCD greatly simplified. In particular, we would like to emphasize that there is essentially no need to use Bessel functions for the computation of the pion form factor. Moreover, since the dominance of the pion zero-mode's contribution (having the longest correlation length), is universal for any finite volume effects on any operators, we expect a wide application of this method. It may be useful for heavier hadron form factors, and simulations in the p regime as well.

The rest of this paper is organized as follows. In Sec. II, we review the ϵ expansion of ChPT and present how to compute the correlators at one-loop level. In Sec. III, we consider the two-point functions to illustrate our new idea. Then, our main result for the pseudoscalar-vector-pseudoscalar three-point functions is presented in Sec. IV, including the NLO effects. In Sec. V, we show how to extract the pion vector form factor, and estimate the remaining finite volume effects numerically : we find that it is a few percent level already at $L = 3$ fm. Summary and conclusion are given in Sec. VI.

II. THE ϵ EXPANSION OF CHPT

In this section, we review the ϵ expansion of ChPT, and show how to perform the one-loop level calculation of the correlators. First, we give the counting rule of the ϵ expansion. Second, we write down the chiral lagrangian with pseudo-scalar and vector source terms, and explain a general procedure to calculate correlators from a partition function. Finally, we give the technical details of this study at the end of this section.

A. The chiral Lagrangian

We consider N_f -flavor ChPT in an Euclidean finite volume $V = TL^3$ with the periodic boundary condition in every direction. The Lagrangian [1, 2] is given by

$$\mathcal{L}_{\text{ChPT}} = \frac{F^2}{4} \text{Tr} \left[(\partial_\mu U(x))^\dagger (\partial_\mu U(x)) \right] - \frac{\Sigma}{2} \text{Tr} \left[\mathcal{M}^\dagger U(x) + U^\dagger(x) \mathcal{M} \right] + \cdots, \quad (2)$$

where $U(x)$ denote the chiral field which is an element of the group $SU(N_f)$. Σ is the chiral condensate and F is the pion decay constant both in the chiral limit. The terms omitted by ellipses are the ones at the higher orders. For simplicity, we take the quark mass matrix \mathcal{M} degenerate and diagonal: $\mathcal{M} = \text{diag}(m, m, m, \cdots)$.

In the ϵ regime [9], the vacuum is not fixed but has non-perturbatively large fluctuations. Namely, the zero-mode of the pions must be integrated exactly. Thus, we separate it from the non-zero momentum modes and parametrize the chiral field as

$$U(x) = U_0 \left(\frac{i\sqrt{2}}{F} \xi(x) \right), \quad U_0 \in SU(N_f), \quad (3)$$

where U_0 denotes the zero-modes. The non-zero momentum mode is decomposed as $\xi(x) = T^a \xi^a(x)$ with $SU(N_f)$ generators T^a , for which we use the normalization of $\text{Tr}[T^a T^b] = \frac{1}{2} \delta^{ab}$. Since the constant modes are separated from $\xi(x)$ fields as U_0 , a constraint

$$\int d^4x \xi(x) = 0, \quad (4)$$

must be satisfied to avoid the double-counting of the zero-modes.

Now, we rewrite the chiral Lagrangian Eq. (2) with the ϵ expansion, whose counting rule is given by

$$U_0 \sim \mathcal{O}(1), \\ \epsilon \sim \partial_\mu \sim \frac{1}{V^{1/4}} \sim m_\pi^{1/2} \sim m^{1/4} \sim \xi(x), \quad (5)$$

as

$$\begin{aligned} \mathcal{L}_{\text{ChPT}} = & -\frac{\Sigma}{2} \text{Tr} [\mathcal{M}^\dagger U_0 + U_0^\dagger \mathcal{M}] + \frac{1}{2} \text{Tr} [\partial_\mu \xi \partial_\mu \xi] (x) \\ & + \frac{\Sigma}{2F^2} \text{Tr} [(\mathcal{M}^\dagger U_0 + U_0^\dagger \mathcal{M}) \xi^2] (x) + \dots \end{aligned} \quad (6)$$

From this Lagrangian, one can recognize that we are treating a hybrid system containing bosonic $\xi(x)$ fields and a matrix U_0 , which are weakly interacting.

For $\xi(x)$ fields, one can perform the Gaussian integrals without difficulty. In this work, we use the correlator in quark-line basis,

$$\langle [\xi(x)]_{ij} [\xi(y)]_{kl} \rangle_\xi = \delta_{il} \delta_{jk} \bar{\Delta}(x-y) - \delta_{ij} \delta_{kl} \frac{1}{N_f} \bar{\Delta}(x-y), \quad (7)$$

where the second term comes from the constraint $\text{Tr} \xi = 0$, and

$$\bar{\Delta}(x) \equiv \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx}}{p^2}, \quad (8)$$

describes the propagation of the massless bosons. Here, the summation is taken over the non-zero 4-momentum $p = 2\pi(n_t/T, n_x/L, n_y/L, n_z/L)$, with integers n_μ , except for $p = (0, 0, 0, 0)$, because of the constraint Eq. (4).

While $\xi(x)$ fields are treated perturbatively, the zero-mode denoted by U_0 has to be non-perturbatively integrated (we will denote it by $\langle \cdots \rangle_{U_0}$). It is known that these matrix integrals are expressed by the Bessel functions [18–20], which is a peculiar feature of the ϵ regime. Historically, this special feature of the ϵ regime is used for extracting the leading LEC's, Σ and F , which are more sensitive to the volume than others. However, for the other LEC's at NLO, we should take a different strategy, or we should remove the contamination from the finite size. In this work on the vector form factor of pions, which is related to L_9 , the U_0 integral plays a less important role.

B. Partition function and correlators

In this subsection, we consider the partition function of ChPT in the ϵ regime and show how to calculate the correlation functions. First, we introduce the relevant source terms to the chiral Lagrangian Eq. (2). Since the Lagrangian is invariant under the chiral rotation,

$$U(x) \rightarrow g_L U(x) g_R^\dagger, \quad g_L, g_R \in SU(N_f), \quad (9)$$

the vector or axial vector operators are given through the Noether's theorem for the vectorlike transformation $g_L = g_R$ and the axial one $g_L = g_R^\dagger$. It is easy to see that adding these operators is equivalent to replacing the derivatives by the “covariant” derivatives:

$$\partial_\mu \rightarrow \nabla_\mu U(x) \equiv \partial_\mu U(x) - i(v_\mu(x) + a_\mu(x))U(x) + iU(x)(v_\mu(x) - a_\mu(x)), \quad (10)$$

where $v_\mu(x)$ and $a_\mu(x)$ denote the vector and axial-vector sources, respectively. Similarly, since the Lagrangian is invariant under the Parity transformation,

$$U(x) \rightarrow U^\dagger(x), \quad x = (t, x, y, z) \rightarrow x = (t, -x, -y, -z), \quad (11)$$

adding a scalar $U(x) + U^\dagger(x)$ and a pseudoscalar $U(x) - U^\dagger(x)$ is absorbed in the mass matrix:

$$\mathcal{M} \rightarrow \mathcal{M}_J \equiv \mathcal{M} + s(x) + ip(x), \quad (12)$$

where $s(x)$ and $p(x)$ denote the scalar and pseudoscalar sources, respectively. we set $s(x) = a_\mu(x) = 0$ in the following.

Next, let us introduce the NLO terms of the chiral lagrangian. However, some of them are irrelevant to our calculations. In this study, it is enough to consider the terms with the low-energy constants L_i ($i = 4, \dots, 9$). Namely, we consider the Lagrangian

$$\begin{aligned}
\mathcal{L}(s, p, v_\mu, a_\mu) = & \frac{F^2}{4} \text{Tr}[\nabla_\mu U^\dagger(x) \nabla_\mu U(x)] - \frac{\Sigma}{2} \text{Tr}[\mathcal{M}_J^\dagger U(x) + U^\dagger(x) \mathcal{M}_U] \\
& + L_4 \frac{2\Sigma}{F^2} \text{Tr}[(\nabla_\mu U(x))^\dagger \nabla_\mu U(x)] \times \text{Tr}[\mathcal{M}_J^\dagger U(x) + U^\dagger(x) \mathcal{M}_J] \\
& + L_5 \frac{2\Sigma}{F^2} \text{Tr}[(\nabla_\mu U(x))^\dagger \nabla_\mu U(x) (\mathcal{M}_J^\dagger U(x) + U^\dagger(x) \mathcal{M}_J)] \\
& - L_6 \left(\frac{2\Sigma}{F^2} \text{Tr}[\mathcal{M}_J^\dagger U(x) + U^\dagger(x) \mathcal{M}_J] \right)^2 \\
& - L_7 \left(\frac{2\Sigma}{F^2} \text{Tr}[\mathcal{M}_J^\dagger U(x) - U^\dagger(x) \mathcal{M}_J] \right)^2 \\
& - L_8 \left(\frac{2\Sigma}{F^2} \right)^2 \text{Tr}[\mathcal{M}_J^\dagger U(x) \mathcal{M}_J^\dagger U(x) + U^\dagger(x) \mathcal{M}_J U^\dagger(x) \mathcal{M}_J] \\
& + iL_9 \text{Tr}[F_{\mu\nu}^R(x) \nabla^\mu U(x) (\nabla^\nu U(x))^\dagger + F_{\mu\nu}^L(x) (\nabla^\mu U(x))^\dagger \nabla^\nu U(x)], \quad (13)
\end{aligned}$$

where

$$\begin{aligned}
F_{\mu\nu}^I(x) &= \partial_\mu F_\nu^I(x) - \partial_\nu F_\mu^I(x) - i[F_\mu^I(x), F_\nu^I(x)], \quad I = R, L, \\
F_\mu^R(x) &= v_\mu(x) + a_\mu(x), \quad F_\mu^L(x) = v_\mu(x) - a_\mu(x). \quad (14)
\end{aligned}$$

The calculation of ChPT is performed in the functional integral formalism. The partition function is defined by

$$\mathcal{Z}(s, p, v_\mu, a_\mu) = \int \prod_x dU(x) \exp \left[- \int d^4x \mathcal{L}(s, p, v_\mu, a_\mu) \right], \quad (15)$$

and the correlators are computed by differentiating it with respect to the corresponding sources, and take their zero limits. The pseudoscalar two-point function, for example, is given by

$$\langle P^a(x) P^b(y) \rangle = \frac{1}{\mathcal{Z}(0, 0, 0, 0)} \frac{\delta}{\delta p^a(x)} \frac{\delta}{\delta p^b(y)} \mathcal{Z}(s, p, v_\mu, a_\mu) \Big|_{s, p, v_\mu, a_\mu = 0}, \quad (16)$$

where $p^a(x)$ denotes the coefficient of an $SU(N_f)$ generator T^a , where we decompose the source as $p(x) = T^a p^a(x)$.

One should note that our non-trivial parametrization of $U(x)$ needs a non-trivial Jacobian in the functional integration measure:

$$\int \prod_x dU(x) = \int dU_0 \prod_x d\xi(x) \mathcal{J}(U_0, \xi). \quad (17)$$

A perturbative calculation [11, 23] has shown

$$\mathcal{J}(U_0, \xi) = \exp \left(- \int d^4x \frac{N_f}{3F^2V} \text{Tr} \xi^2(x) + \mathcal{O}(\epsilon^4) \right), \quad (18)$$

which can be regarded as an additional mass term of the $\xi(x)$ fields at the one-loop level. Note that this additional mass doesn't vanish even in the $m \rightarrow 0$ limit, which keeps the theory infra-red finite.

Finally, let us consider the θ vacuum and fixing topology. In the ϵ regime, we often consider a fixed topological sector, rather than the full QCD vacuum with the vacuum angle $\theta = 0$. For this purpose, we encode the non-zero vacuum angle θ to the mass term [24],

$$\mathcal{M} \rightarrow \mathcal{M}_\theta = \mathcal{M} \exp(-i\theta/N_f), \quad (19)$$

using the axial $U(1)_A$ rotation. Then we can perform a Fourier transformation with respect to θ to obtain the partition function at fixed topology,

$$\mathcal{Z}^Q(s, p, v_\mu, a_\mu) \equiv \int_0^{2\pi} \frac{d\theta}{2\pi} \left[e^{i\theta Q} \mathcal{Z}(s, p, v_\mu, a_\mu) \Big|_{\mathcal{M}=\mathcal{M}_\theta} \right], \quad (20)$$

where Q denotes the topological charge of the original gauge fields. It is known that this θ integral can be absorbed in the group integration of the zero-mode: redefining the zero mode,

$$e^{i\theta/N_f} U(x) = \overline{U}_0 \exp \left(\frac{i\sqrt{2}}{F} \xi(x) \right), \quad (21)$$

where $\overline{U}_0 \in U(N_f)$, the zero-mode part of the functional integral is modified to

$$\int \frac{d\theta}{2\pi} \exp(i\theta Q) \int_{SU(N_f)} dU_0 F(\mathcal{M}^\dagger e^{i\theta/N_f} U_0) = \int_{U(N_f)} d\overline{U}_0 (\det \overline{U}_0)^Q F(\mathcal{M}^\dagger \overline{U}_0), \quad (22)$$

where we have used the fact that the zero-mode in the Lagrangian always appears as a function of $\mathcal{M}^\dagger e^{i\theta/N_f} U_0$ (and its Hermitian conjugate). Fixing the topology is technically easier since the $U(N_f)$ group integral is simpler than that of $SU(N_f)$. It is also useful for investigating the finite volume physics which is sensitive to the topology of the gauge fields. It is important to note that the fixing topology effect is totally encoded in the pion-zero mode, and therefore, is automatically eliminated once the effect of the latter is eliminated. Since we will be able to cancel the effect of U_0 (from the LO contribution), in the following sections, we don't distinguish U_0 and \overline{U}_0 unless explicitly stated.

We are now ready for the 1-loop computations. However, we would like to give some useful technical details which simplify the calculations, in the next subsection.

C. Technical details

Because of the non-trivial parametrization of the chiral field, we have a lot of diagrams to be computed in the ϵ expansion of ChPT even at NLO. Here we rewrite the Lagrangian using the non-self-contracting (NSC) vertices, and compute some of 1-loop diagrams in advance, as corrections to the chiral Lagrangian. This reduces the number of diagrams and simplify our calculation.

The n -point NSC vertex is defined by

$$[\xi^n(x)]^{NSC} \equiv \xi^n(x) - (\text{all possible } \xi \text{ contractions}). \quad (23)$$

and we can absorb the contracted part in the redefinition of the lower dimensional terms in the Lagrangian. Note that $\langle [\xi^n(x)]^{NSC} \rangle_\xi = 0$ by definition. For example, a term in the Lagrangian at NLO can be re-expressed by

$$\begin{aligned} \frac{1}{6F^2} \text{Tr}[\partial_\mu \xi \xi \partial_\mu \xi \xi - \xi^2 (\partial_\mu \xi)^2] &= \frac{1}{6F^2} \text{Tr}[\partial_\mu \xi \xi \partial_\mu \xi \xi - \xi^2 (\partial_\mu \xi)^2]^{NSC} \\ &+ \frac{1}{2} \text{Tr}[(\partial_\mu \xi)^2] \Delta Z^\xi + \frac{1}{2} \text{Tr}[\xi^2] \Delta M^2, \end{aligned} \quad (24)$$

where

$$\Delta M^2 = -\frac{N_f}{3F^2} \partial_\mu^2 \bar{\Delta}(0) = \frac{N_f}{3F^2 V}, \quad (25)$$

can be absorbed in the re-definition of the mass term, and

$$\Delta Z^\xi = -\frac{N_f}{3F^2} \bar{\Delta}(0), \quad (26)$$

can be absorbed in the re-definition of the kinetic term. Here, and in the following, the momentum summations in embedded in $\bar{\Delta}(0)$ etc. are kept unperformed until the very end of the calculation, except for the trivially clear cases like $\partial_\mu^2 \bar{\Delta}(0) = -1/V$, $\partial_\mu \bar{\Delta}(0) = 0$. In this work, we employ the dimensional regularization for the loop integrals.

With the NSC vertices, the action is expanded as

$$\mathcal{S}_{\text{ChPT}} = \int d^4x \mathcal{L} = \mathcal{S}^{\text{LO}} + \mathcal{S}^{\text{NLO}} + \mathcal{S}^{\text{src}} + \dots, \quad (27)$$

where

$$\mathcal{S}^{\text{LO}} = -\frac{Z^\Sigma \Sigma V}{2} \text{Tr}[\mathcal{M}^\dagger U_0 + U_0^\dagger \mathcal{M}] + \int d^4x \left\{ \frac{1}{2} \text{Tr}[\partial_\mu \xi \partial^\mu \xi](x) \right\} (Z^\xi)^2, \quad (28)$$

$$\mathcal{S}^{\text{NLO}} = \mathcal{S}_K^{\text{NLO}} + \mathcal{S}_M^{\text{NLO}},$$

$$\mathcal{S}^{\text{src}} = \int d^4x \text{Tr}[p(x)P(x) + v_\mu(x)V^\mu(x)], \quad (29)$$

where

$$\mathcal{S}_K^{\text{NLO}} \equiv \int d^4x \frac{1}{6F^2} \text{Tr}[\partial_\mu \xi \xi \partial^\mu \xi \xi - \xi^2 \partial_\mu \xi \partial^\mu \xi]^{NSC}(x), \quad (30)$$

$$\mathcal{S}_M^{\text{NLO}} \equiv \int d^4x \frac{\Sigma}{2F^2} \text{Tr} \left[\left(\mathcal{M}^\dagger U_0 + U_0^\dagger \mathcal{M} + \frac{N_f}{\Sigma V} \right) \xi^2 \right]^{NSC}(x). \quad (31)$$

Note that the linear term in $\xi(x)$ disappears because of the constraint Eq. (4).

Here, the source operators are given by

$$\begin{aligned} P(x) &= iZ^{P1} \frac{\Sigma}{2} [U_0 - U_0^\dagger] - Z^{P2} \frac{\Sigma}{\sqrt{2}F} [U_0 \xi + \xi U_0^\dagger] - Z^{P3} \frac{i\Sigma}{2F^2} [U_0 \xi^2 - \xi^2 U_0^\dagger]^{NSC} \\ &\quad + \frac{i\Sigma}{12F^4} \bar{\Delta}(0) [U_0 - U_0^\dagger] \text{Tr}[\xi^2]^{NSC} \\ &\quad + \frac{\Sigma}{3\sqrt{2}F^3} [U_0 \xi^3 + \xi^3 U_0^\dagger]^{NSC} + \frac{i\Sigma}{12F^4} [U_0 \xi^4 - \xi^4 U_0^\dagger]^{NSC} \\ &\quad - iL_4 \frac{4\Sigma}{F^4} \left(\text{Tr}[\partial_\mu \xi \partial^\mu \xi]^{NSC} \right) \times [U_0 - U_0^\dagger] - iL_5 \frac{4\Sigma}{F^4} [U_0 \partial_\mu \xi \partial^\mu \xi - \partial_\mu \xi \partial^\mu \xi U_0^\dagger]^{NSC} \\ &\quad + \mathcal{O}(\epsilon^5), \quad (32) \\ V^\mu(x) &= -\frac{FZ^{V1}}{\sqrt{2}} [U_0 \partial^\mu \xi U_0^\dagger - \partial^\mu \xi] \\ &\quad + \frac{iZ^{V2}}{2} [U_0 (\partial^\mu \xi \xi - \xi \partial^\mu \xi) U_0^\dagger + (\partial^\mu \xi \xi - \xi \partial^\mu \xi)]^{NSC} \\ &\quad + \frac{1}{3\sqrt{2}F} [U_0 (\partial^\mu \xi \xi^2 - 2\xi \partial^\mu \xi \xi + \xi^2 \partial^\mu \xi) U_0^\dagger - (\partial^\mu \xi \xi^2 - 2\xi \partial^\mu \xi \xi + \xi^2 \partial^\mu \xi)]^{NSC} \\ &\quad - \frac{i}{12F^2} [U_0 (\partial^\mu \xi \xi^3 - 3\xi \partial^\mu \xi \xi^2 + 3\xi^2 \partial^\mu \xi \xi - \xi^3 \partial^\mu \xi) U_0^\dagger \\ &\quad \quad + (\partial^\mu \xi \xi^3 - 3\xi \partial^\mu \xi \xi^2 + 3\xi^2 \partial^\mu \xi \xi - \xi^3 \partial^\mu \xi)]^{NSC} \\ &\quad - \frac{2iL_9}{F^2} \partial_\nu [U_0 (\partial^\nu \xi \partial^\mu \xi - \partial^\mu \xi \partial^\nu \xi) U_0^\dagger + (\partial^\nu \xi \partial^\mu \xi - \partial^\mu \xi \partial^\nu \xi)]^{NSC} + \mathcal{O}(\epsilon^6), \quad (33) \end{aligned}$$

where

$$Z^\Sigma = 1 - \frac{N_f^2 - 1}{N_f F^2} \bar{\Delta}(0), \quad (34)$$

$$Z^\xi = 1 - \frac{N_f}{6F^2} \bar{\Delta}(0), \quad (35)$$

$$Z^{P1} = Z^\Sigma + \mathcal{O}(\epsilon^4), \quad (36)$$

$$Z^{P2} = 1 - \frac{2N_f^2 - 3}{3N_f F^2} \bar{\Delta}(0), \quad (37)$$

$$Z^{P3} = 1 - \frac{N_f}{2F^2} \bar{\Delta}(0) + \frac{1}{N_f F^2} \bar{\Delta}(0), \quad (38)$$

$$Z^{V1} = 1 - \frac{2N_f}{3F^2} \bar{\Delta}(0), \quad (39)$$

$$Z^{V2} = 1 - \frac{5N_f}{6F^2} \bar{\Delta}(0). \quad (40)$$

In the above expression, the argument (x) of $\xi(x)$ is omitted for simplicity. In this work, we don't consider contact correlators at the same position, such as $\langle P(x)V(x) \rangle$. We have, therefore, only collected the terms linear in the sources $p(x)$ and $v_\mu(x)$.

Here we note that except for Z^{V^2} , we can absorb all the Z factors into the redefinition of the wave functions (ξ fields), or the coupling constants, by defining

$$\xi'(x) \equiv Z^\xi \xi(x), \quad (41)$$

$$\Sigma_{\text{eff}} \equiv Z^\Sigma \Sigma, \quad (42)$$

$$F_{\text{eff}} \equiv F \left(1 - \frac{N_f}{2F^2} \bar{\Delta}(0) \right). \quad (43)$$

Therefore, except for the 4-th term in Eq. (32), the vertex corrections of the two-point and three-point correlators can be obtained by simply replacing the coefficients of the LO results with the shifted ones Σ_{eff} and F_{eff} , except for multiplying the coefficient of the second term in $V^\mu(x)$,

$$Z^{V^2}/(Z^\xi)^2 = 1 - \frac{N_f}{2F^2} \bar{\Delta}(0), \quad (44)$$

and the third term in $P(x)$,

$$Z^{P3'} \equiv \frac{Z^{P3}}{Z^\Sigma (Z^\xi)^2} \left(\frac{F_{\text{eff}}}{F} \right)^2 = 1 - \frac{N_f}{6F^2} \bar{\Delta}(0). \quad (45)$$

With this action, for any operator O (as a function of ξ and U_0) in the ϵ expansion,

$$O = O^{\text{LO}} + O^{\text{NLO}} + \dots, \quad (46)$$

its expectation value is perturbatively evaluated as,

$$\begin{aligned} \langle O \rangle &\equiv \frac{\int \mathcal{D}U_0 \mathcal{D}\xi \left[(O^{\text{LO}} + O^{\text{NLO}} + \dots) e^{-S^{\text{LO}} - S^{\text{NLO}} + \dots} \right]}{\int \mathcal{D}U_0 \mathcal{D}\xi \left[e^{-S^{\text{LO}} - S^{\text{NLO}} + \dots} \right]} \\ &= \langle \langle O^{\text{LO}} \rangle_\xi \rangle_{U_0} + \left[\langle \langle O^{\text{NLO}} \rangle_\xi \rangle_{U_0} - \langle \langle O^{\text{LO}} S^{\text{NLO}} \rangle_\xi \rangle_{U_0} + \langle \langle O^{\text{LO}} \rangle_\xi \rangle_{U_0} \langle \langle S^{\text{NLO}} \rangle_\xi \rangle_{U_0} \right] + \dots, \end{aligned} \quad (47)$$

where we have used the following notations,

$$\langle O_1(U_0) \rangle_{U_0} \equiv \frac{\int \mathcal{D}U_0 e^{\frac{\Sigma_{\text{eff}} V}{2} \text{Tr}[\mathcal{M}^\dagger U_0 + U_0^\dagger \mathcal{M}]} O_1(U_0)}{\int \mathcal{D}U_0 e^{\frac{\Sigma_{\text{eff}} V}{2} \text{Tr}[\mathcal{M}^\dagger U_0 + U_0^\dagger \mathcal{M}]}}, \quad (48)$$

$$\langle O_2(\xi) \rangle_\xi \equiv \frac{\int \mathcal{D}\xi e^{-\int d^4x \frac{1}{2} \text{Tr}[\xi(-\partial_\mu^2)\xi](x)} O_2(\xi)}{\int \mathcal{D}\xi e^{-\int d^4x \frac{1}{2} \text{Tr}[\xi(-\partial_\mu^2)\xi](x)}}. \quad (49)$$

Note that, due to the use of NSC vertices, we don't need to calculate the fourth term in Eq. (47) since $\langle \mathcal{S}^{\text{NLO}} \rangle_\xi = 0$.

In the usual $\theta = 0$ vacuum, $\mathcal{D}U_0$ denotes a Haar measure on $SU(N_f)$, while it should be replaced by $\mathcal{D}U_0(\det U_0)^Q$ on $U(N_f)$, for a fixed topological sector as discussed in the previous subsection.

III. TWO-POINT FUNCTIONS

As we have mentioned in Sec. I, the dominant finite volume effect on correlators comes from the pion zero-mode. Since the zero-mode itself does not depend on the space-time position x , its effect always appear as an x -independent constant term or overall constants of x -dependent terms. In either case, it is not difficult to eliminate these zero-mode's effects from the correlators. In this section, we demonstrate this new idea taking the two-point pseudoscalar correlators, as an easiest example.

A. LO calculation

Let us consider a pseudoscalar operator in the charged pion channel,

$$P^1(x) \equiv \frac{1}{2} ([P(x)]_{12} + [P(x)]_{21}). \quad (50)$$

From the chiral symmetry, it is easy to confirm that its two-point function satisfies

$$\langle [P(x)]_{12} [P(y)]_{12} \rangle = \langle [P(x)]_{21} [P(y)]_{21} \rangle = 0, \quad (51)$$

and

$$\langle [P(x)]_{12} [P(y)]_{21} \rangle = \langle [P(x)]_{21} [P(y)]_{12} \rangle = 2\langle P^1(x) P^1(y) \rangle. \quad (52)$$

The quark field basis $[P(x)]_{ij}$ is convenient unless we consider the neutral sector of ChPT, since $\langle [P(x)]_{ij} [P(y)]_{ji} \rangle$ shares the same normalization of the so-called “connected” contribution of the conventional meson correlators in lattice QCD. Therefore, we use $[P(x)]_{ij}$ rather than the original $P^1(x)$ in the following analysis.

Now we can write down the two-point function to $\mathcal{O}(\epsilon^2)$,

$$\langle [P(x)]_{12} [P(y)]_{21} \rangle = -\frac{\Sigma_{\text{eff}}^2}{4} \langle \mathcal{A}(U_0) \rangle_{U_0} + \frac{\Sigma_{\text{eff}}^2}{2F_{\text{eff}}^2 V} \langle \mathcal{B}(U_0) \rangle_{U_0} \sum_{p \neq 0} \frac{e^{ip(x-y)}}{p^2}, \quad (53)$$

where

$$\mathcal{A}(U_0) = [U_0 - U_0^\dagger]_{12}[U_0 - U_0^\dagger]_{21} + \frac{1}{2}([U_0 - U_0^\dagger]_{12})^2 + \frac{1}{2}([U_0 - U_0^\dagger]_{21})^2, \quad (54)$$

$$\mathcal{B}(U_0) = 2 + [U_0]_{11}[U_0]_{22} + [U_0^\dagger]_{11}[U_0^\dagger]_{22} - ([U_0]_{12} + [U_0^\dagger]_{12})([U_0]_{21} + [U_0^\dagger]_{21})/N_f. \quad (55)$$

Note that some NLO contribution is already involved in Σ_{eff} or F_{eff} since we have resummed the Lagrangian with NSC vertices.

This correlator in Eq. (53) is a known result in the literature, and one can find how to evaluate $\langle \mathcal{A}(U_0) \rangle_{U_0}$ and $\langle \mathcal{B}(U_0) \rangle_{U_0}$ in, for example, Ref. [15]. In particular, the x and y -independent constant term is known as a special feature of the ϵ regime, and can be used for extracting Σ . In this work, however, we will eliminate this constant term in the end of the calculation. Therefore, we have to treat the second term of Eq. (53) as the LO contribution, and the calculation at one order higher is needed.

B. NLO calculation

Next, let us compute the NLO contribution. Here and in the following, we simply neglect the contribution to the constant part.

For the third term of Eq. (47), we have

$$\begin{aligned} -\langle \langle [P(x)]_{12} [P(y)]_{21} \rangle^{\text{LO}} \mathcal{S}^{\text{NLO}} \rangle_{\xi} \rangle_{U_0} &= -\langle \langle [P(x)]_{12} [P(y)]_{21} \rangle^{\text{LO}} \mathcal{S}_M^{\text{NLO}} \rangle_{\xi} \rangle_{U_0} \\ &= \frac{\Sigma_{\text{eff}}^2}{2F_{\text{eff}}^2 V} \langle \mathcal{D}(U_0) \rangle_{U_0} (-M_{12}^2) \sum_{p \neq 0} \frac{e^{ip(x-y)}}{(p^2)^2}, \end{aligned} \quad (56)$$

where $M_{12}^2 \equiv (m_1 + m_2)\Sigma_{\text{eff}}/F^2$, and the dimensionless U_0 integral part is given by

$$\mathcal{D}(U_0) = \sum_{k=0}^4 \mathcal{D}^k(U_0), \quad (57)$$

$$\mathcal{D}^0(U_0) = [U_0]_{11} + [U_0]_{22} + [U_0^\dagger]_{11} + [U_0^\dagger]_{22}, \quad (58)$$

$$\mathcal{D}^1(U_0) = \frac{N_f}{\mu_1 + \mu_2} (2 - [U_0]_{11}[U_0]_{22} - [U_0^\dagger]_{11}[U_0^\dagger]_{22}), \quad (59)$$

$$\mathcal{D}^2(U_0) = \sum_{i,j} \frac{\delta_{i1}\delta_{2j} + \delta_{i2}\delta_{1j}}{2} \left[[U_0]_{ii} \left(\frac{[U_0 \mathcal{M}^\dagger U_0]_{jj} - m_j}{m_1 + m_2} + \frac{2N_f}{\mu_1 + \mu_2} [U_0]_{jj} \right) + h.c. \right], \quad (60)$$

$$\begin{aligned} \mathcal{D}^3(U_0) &= -\sum_{i,j} \frac{\delta_{i1}\delta_{2j} + \delta_{i2}\delta_{1j}}{N_f} ([U_0]_{ij} + [U_0^\dagger]_{ij}) \\ &\quad \times \left[\frac{[U_0 \mathcal{M}^\dagger U_0]_{ji} + [U_0^\dagger \mathcal{M} U_0^\dagger]_{ji}}{m_1 + m_2} + \frac{N_f}{\mu_1 + \mu_2} ([U_0]_{ji} + [U_0^\dagger]_{ji}) \right], \end{aligned} \quad (61)$$

$$\mathcal{D}^4(U_0) = \frac{([U_0]_{12} + [U_0^\dagger]_{12})([U_0]_{21} + [U_0^\dagger]_{21})}{N_f} \left(\frac{1}{N_f} \sum_i^{N_f} \frac{m_i([U_0]_{ii} + [U_0^\dagger]_{ii})}{m_1 + m_2} + \frac{N_f}{\mu_1 + \mu_2} \right), \quad (62)$$

where $\mu_i = m_i \Sigma_{\text{eff}} V$. Here we have given more general results than our set-up in this work: with non-degenerate N_f -flavor quark masses m_i 's. The degenerate results can be obtained simply taking $m_i \rightarrow m$ in the above formulas. Note that we have neglected trivially vanishing matrix elements like $\langle [U_0]_{ij} \rangle_{U_0} = 0$ for $i \neq j$.

The second term of Eq. (47) is given by

$$\langle \langle [[P(x)]_{12} [P(y)]_{21}]^{\text{NLO}} \rangle_\xi \rangle_{U_0} = -\frac{\Sigma_{\text{eff}}^2}{4F_{\text{eff}}^4 V^2} \langle \mathcal{C}(U_0) \rangle_{U_0} \sum_{p_1 \neq 0} \sum_{p_2 \neq 0} \frac{e^{ip_1(x-y)}}{p_1^2} \frac{e^{ip_2(x-y)}}{p_2^2}, \quad (63)$$

where

$$\begin{aligned} \mathcal{C}(U_0) = & \left(\frac{4}{N_f} - N_f \right) \left(2 - [U_0]_{11}[U_0]_{22} - [U_0^\dagger]_{11}[U_0^\dagger]_{22} \right) \\ & + \left(1 + \frac{2}{N_f^2} \right) ([U_0]_{12} - [U_0^\dagger]_{12}) ([U_0]_{21} - [U_0^\dagger]_{21}). \end{aligned} \quad (64)$$

To summarize our results, it is useful to define the “massive” propagator,

$$\bar{\Delta}(x; M^2) \equiv \frac{1}{V} \sum_{p \neq 0} \frac{e^{ipx}}{p^2 + M^2}, \quad (65)$$

and noting for $M \sim \mathcal{O}(\epsilon^2)$,

$$\frac{1}{p^2} - M^2 \frac{1}{(p^2)^2} = \frac{1}{p^2 + M^2} + \mathcal{O}(M^4), \quad (66)$$

the correlator in a simple form is obtained,

$$\begin{aligned} \langle [P(x)]_{12} [P(y)]_{21} \rangle = & \text{const.} + \frac{\Sigma_{\text{eff}}^2}{2F_{\text{eff}}^2 V} \langle \mathcal{B}(U_0) \rangle_{U_0} \sum_{p \neq 0} \frac{e^{ip(x-y)}}{p^2 + M_{12}^2 Z_M^{2\text{pt}}} \\ & - \frac{\Sigma_{\text{eff}}^2}{4F_{\text{eff}}^4 V^2} \langle \mathcal{C}(U_0) \rangle_{U_0} \sum_{p_1 \neq 0} \sum_{p_2 \neq 0} \frac{e^{ip_1(x-y)}}{p_1^2} \frac{e^{ip_2(x-y)}}{p_2^2}, \end{aligned} \quad (67)$$

where *const.* denotes the constant term we have omitted, and

$$Z_M^{2\text{pt}} = 1 + \frac{\langle \mathcal{D}(U_0) - \mathcal{B}(U_0) \rangle_{U_0}}{\langle \mathcal{B}(U_0) \rangle_{U_0}}. \quad (68)$$

The second term of Eq. (67) looks almost the same as the conventional massive pion propagator in the p regime. In fact, we can smoothly obtain this by taking the $V \rightarrow \infty$ limit where one obtains $\langle \mathcal{B}(U_0) \rangle_{U_0} \rightarrow 4$, $\langle \mathcal{A}(U_0) \rangle_{U_0} = \langle \mathcal{C}(U_0) \rangle_{U_0} \rightarrow 0$, and $Z_M^{2\text{pt}} \rightarrow 1$.

The third term of Eq. (67) is another peculiar term in the ϵ regime, which originally comes from a 3-pion state, consisting of one having zero momentum and two having non-zero momenta. At this order, it looks a propagation of two massless particles. However, these propagators should have mass corrections at higher orders, at least, the one from the measure term in Eq. (18). We expect that it cannot reach a long-distance, compared to the single particle propagation. In the following analysis, we simply neglect this NLO term and similar terms in the three-point functions. This truncation may be numerically justified by carefully checking the plateau of the effective mass, when we simulate lattice QCD [6].

C. Removing dominant finite volume effects in the ϵ expansion

Now we are ready to cancel the dominant volume effects. First, we insert spacial momentum to the operators. Namely, we consider

$$\begin{aligned} C_{PP}^{2\text{pt}}(t, \mathbf{p}) &\equiv \langle [P(x_0; \mathbf{p})]_{12} [P(y_0; -\mathbf{p})]_{21} \rangle, \\ [P(x_0, \mathbf{p})]_{ij} &\equiv \int d^3x e^{-i\mathbf{p}\cdot\mathbf{x}} [P(x)]_{ij}, \end{aligned} \quad (69)$$

where x_0 is the temporal element of x , $t = x_0 - y_0$, and $\mathbf{p} = 2\pi(n_x, n_y, n_z)/L$ is the 3-dimensional momentum. Then, the unwanted constant contribution *consts.* automatically disappears for $\mathbf{p} \neq \mathbf{0}$. It is also intuitively reasonable that the higher energy states having momenta are less sensitive to the finite volume effects. Even in the case of $\mathbf{p} = \mathbf{0}$, it vanishes in a simple subtraction with respect to time: $\Delta_t[P(t, \mathbf{p})]_{ij} \equiv [P(t, \mathbf{p})]_{ij} - [P(t_{\text{ref}}, \mathbf{p})]_{ij}$ with a reference time-slice t_{ref} .

The second step is to take a ratio of the correlators with different momenta. For example, by shifting $y_0 \rightarrow 0$, and renaming $x_0 = t$, we have

$$\begin{aligned} R^{2\text{pt}}(t; \mathbf{p}) &\equiv \frac{\langle [P(t; \mathbf{p})]_{12} [P(0; -\mathbf{p})]_{21} \rangle}{\langle \Delta_t[P(t; \mathbf{0})]_{12} [P(0; \mathbf{0})]_{21} \rangle} \\ &= \frac{E^{2\text{pt}}(\mathbf{0}) \sinh(E^{2\text{pt}}(\mathbf{0})T/2)}{E^{2\text{pt}}(\mathbf{p}) \sinh(E^{2\text{pt}}(\mathbf{p})T/2)} \times \frac{\cosh(E^{2\text{pt}}(\mathbf{p})(t - T/2))}{\cosh(E^{2\text{pt}}(\mathbf{0})(t - T/2)) - \cosh(E^{2\text{pt}}(\mathbf{0})(t_{\text{ref}} - T/2))}, \end{aligned} \quad (70)$$

where

$$E^{2\text{pt}}(\mathbf{p}) \equiv \sqrt{M_{12}^2 Z_M^{2\text{pt}} + \mathbf{p}^2}. \quad (71)$$

The ratio $R^{2\text{pt}}(t; \mathbf{p})$ is no more dependent on $\langle \mathcal{A}(U_0) \rangle_{U_0}$ or $\langle \mathcal{B}(U_0) \rangle_{U_0}$. In fact, this expression is exactly the same as the same ratio in the p expansion, except for the mass renormalization factor $Z_M^{2\text{pt}}$. Namely, we have minimized the features of the ϵ regime in the two-point correlator. It is also important to note that $R^{2\text{pt}}(t; \mathbf{p})$ is finite even in the limit of $E^{2\text{pt}}(\mathbf{0}) \rightarrow 0$.

Since the above ratio $R^{2\text{pt}}(t; \mathbf{p})$ has no dependence on LEC's of ChPT, it is not phenomenologically interesting. However, it is a good test quantity for lattice QCD to check the validity of the above arguments. Recently, JLQCD collaboration [6] compared the ratio $R^{2\text{pt}}(t; \mathbf{p})$ to the numerical data in the both cases with $M_{12}\sqrt{Z_M^{2\text{pt}}} = 0$ and 100 MeV and found a fairly good agreement, which suggests that the NLO corrections in $\sqrt{Z_M^{2\text{pt}}}$ and the third term of Eq. (67) we have neglected are actually small.

Since the x -independence of the pion zero-mode and its dominance in the finite volume effects are universal and true in any correlation functions at any sizes of the volume, we expect wide applications of our method. Namely, inserting momenta to the correlators and taking a ratios of them generally makes a less sensitive quantity to the volume than the original ones. We will see this is true for the three-point functions in the next section.

IV. THREE-POINT FUNCTION

In this section, we calculate our main target, the pseudoscalar-vector-pseudoscalar three-point function in a finite volume in the ϵ expansion of ChPT, which is relevant for extracting the vector pion form factor. However, we should note that the pion form factor itself is not a quantity described within ChPT alone. In numerical studies [21, 22] it is known that the vector meson largely contributes to the results, which cannot be explained by ChPT. Even in such a case, we still expect that the correction from the finite volume can be treated within ChPT, as the heavier hadrons, including the vector mesons, do not propagate very long. Therefore, in this section, we compute the finite volume effects on the three-point function within the ϵ expansion of ChPT. Once the main part of finite volume effects are removed, the remaining pion form factor should include the physics beyond ChPT.

A. Three-point functions and form factors

First, we briefly review how the three-point functions are related to the pion form factors. Our main target in this work is the vector form factor, defined by

$$\langle \pi^a(p_2) | V_\mu^b(x) | \pi^c(p_1) \rangle = i\epsilon^{abc}(p_1 + p_2)_\mu F_V(t), \quad (72)$$

where $|\pi^a(p)\rangle$ denotes the on-shell pion state with momentum p , $V_\mu^b(x)$ is the coefficient of an $SU(2)$ generator τ^b in the vector operator, and $t = (p_1 - p_2)^2$.

For lattice QCD calculations, it is convenient to take $b = 3$ component

$$V_\mu^3(x) = \frac{1}{2}\bar{u}\gamma_\mu u(x) - \frac{1}{2}\bar{d}\gamma_\mu d(x). \quad (73)$$

Using a conventional notation

$$|\pi^1(p)\rangle = \frac{|\pi^+(p)\rangle + |\pi^-(p)\rangle}{\sqrt{2}}, \quad |\pi^2(p)\rangle = \frac{|\pi^+(p)\rangle - |\pi^-(p)\rangle}{\sqrt{2}i}, \quad (74)$$

where $|\pi^\pm(p)\rangle$ denotes the charged pion state, and iso-spin symmetry (we assume $m_u = m_d = m$),

$$\langle \pi^+(p_2) | V_\mu^3(x) | \pi^+(p_1) \rangle = -\langle \pi^-(p_2) | V_\mu^3(x) | \pi^-(p_1) \rangle, \quad (75)$$

as well as the electric charge conservation,

$$\langle \pi^+(p_2) | V_\mu^3(x) | \pi^-(p_1) \rangle = \langle \pi^-(p_2) | V_\mu^3(x) | \pi^+(p_1) \rangle = 0, \quad (76)$$

one obtains a simpler formula,

$$\langle \pi^+(p_2) | V_\mu^3(x) | \pi^+(p_1) \rangle = (p_1 + p_2)_\mu F_V(t). \quad (77)$$

It is also important to note for the isospin zero current,

$$V_\mu^0(x) = \bar{u}\gamma_\mu u(x) + \bar{d}\gamma_\mu d(x), \quad (78)$$

that its form factor is zero:

$$\langle \pi^a(p_2) | V_\mu^0(x) | \pi^b(p_1) \rangle = 0 \quad \text{for any } a, b, \quad (79)$$

since the pions have zero Baryon charge. In ChPT, this situation is more directly shown by $V_\mu^0(x) = \text{Tr } V_\mu(x) = 0$ in Eq. (33). Namely, there exists no corresponding current within ChPT. Therefore, for the electro-magnetic current defined by

$$J_\mu^{EM} \equiv V_\mu^3(x) + \frac{1}{6}V_\mu^0(x) = \frac{2}{3}\bar{u}\gamma_\mu u(x) - \frac{1}{3}\bar{d}\gamma_\mu d(x), \quad (80)$$

one can show an identity,

$$\langle \pi^+(p_2) | V_\mu^3(x) | \pi^+(p_1) \rangle = \langle \pi^+(p_2) | J_\mu^{EM}(x) | \pi^+(p_1) \rangle. \quad (81)$$

Namely, we don't have to distinguish the vector form factor from the electro-magnetic form factor of the pions.

In the literature, the finite volume correction on the hadronic matrix elements is often computed by just replacing the quantum loop momentum integrals by a discrete summation. However, in such a calculation, one assumes that one can apply the same LSZ reduction formula as in the $V \rightarrow \infty$ limit, to relate the form factor to the three-point function,

$$\begin{aligned} \int d^4x e^{ip_2x} \int d^4z e^{-ip_1z} \langle [P(x)]_{12} V_\mu^3(y) [P(z)]_{21} \rangle \\ = \frac{\langle 0 | [P(0)]_{12} | \pi^+(p_2) \rangle \langle \pi^+(p_1) | [P(0)]_{21} | 0 \rangle}{(p_1^2 + m_\pi^2)(p_2^2 + m_\pi^2)} \langle \pi^+(p_2) | V_\mu^3(y) | \pi^+(p_1) \rangle \end{aligned} \quad (82)$$

In a finite volume (simulated on the lattice), this relation is non-trivial, and one may overlook finite volume corrections to the reduction formula itself. In this work, we work on the finite volume correction within ChPT to

$$\langle [P(x)]_{12} [V_\mu(y)]_{ii} [P(z)]_{21} \rangle, \quad (83)$$

with a general flavor index i . We will soon see that $\langle [P(x)]_{12} [V_\mu(y)]_{ii} [P(z)]_{21} \rangle = (\delta_{i1} - \delta_{i2}) \langle [P(x)]_{12} V_\mu^3(y) [P(z)]_{21} \rangle$. We then perform its Fourier transformation with non-zero momenta, and show how to disentangle the pion form factor from the correlators.

B. LO contribution

In the following, we assume $x_0 > y_0 > z_0$, and denote $t = x_0 - y_0$, $t' = y_0 - z_0$. We further assume that $t, t', t + t' < T/2$ to suppress the effect of modes wrapping around our periodic lattice. It is straightforward to compute the LO contribution to the three-point function in the same way as the two-point function,

$$\begin{aligned} \langle [P(x)]_{12} [V_\mu(y)]_{ii} [P(z)]_{21} \rangle &= (\delta_{i2} - \delta_{i1}) \left(\frac{Z^{V2}}{(Z^\xi)^2} \right) \frac{i\Sigma_{\text{eff}}^2}{4F_{\text{eff}}^2 V^2} \langle \mathcal{E}(U_0) \rangle_{U_0} \\ &\times \sum_{p_1 \neq 0} \sum_{p_2 \neq 0} \frac{-ip_1^\mu - ip_2^\mu}{p_1^2 p_2^2} e^{ip_1(x-y)} e^{ip_2(y-z)}, \end{aligned} \quad (84)$$

where

$$\begin{aligned}\mathcal{E}(U_0) = & \left(2 + 2[U_0]_{11}[U_0]_{22} + 2[U_0^\dagger]_{11}[U_0^\dagger]_{22} \right. \\ & \left. + [U_0]_{11}[U_0^\dagger]_{11} + [U_0]_{22}[U_0^\dagger]_{22} - [U_0]_{12}[U_0^\dagger]_{21} - [U_0]_{21}[U_0^\dagger]_{12} \right). \end{aligned} \quad (85)$$

Here, we have neglected the t and t' independent terms since we will automatically cancel them in the end of our computation.

We have also neglected diagrams where ξ 's are connected in unusual orders, like $x-z-y$ or $z-x-y$, expecting the long propagation between x and z to be exponentially suppressed. This expectation is not true for the zero-momentum contribution at LO. However, as mentioned in the previous section, it is reasonable to expect that the NLO corrections give a “mass” to the correlators and make long-range correlation suppressed compared to the main result. One should be able to numerically check this expectation, since if the neglected contribution is big, it should be detected as unexpected $|x-z|$ dependence.

C. NLO contribution

Next, let us calculate the NLO corrections to the three-point function. As seen in the two-point function, the contribution from $\mathcal{S}_M^{\text{NLO}}$ can be encoded as the mass corrections: together with the LO contribution, one can express it as

$$\begin{aligned} & \langle \langle [P(x)]_{12} [V_\mu(y)]_{ii} [P(z)]_{21} \rangle_{\xi}^{\text{LO}} (1 - \mathcal{S}_M^{\text{NLO}}) \rangle_{U_0} \\ &= (\delta_{i2} - \delta_{i1}) \frac{i\Sigma_{\text{eff}}^2}{4F_{\text{eff}}^2 V^2} \langle \mathcal{E}(U_0) \rangle_{U_0} \sum_{p_1 \neq 0} \sum_{p_2 \neq 0} \frac{-ip_1^\mu - ip_2^\mu}{(p_1^2 + M_{12}^2 Z_M^{\text{3pt}})(p_2^2 + M_{12}^2 Z_M^{\text{3pt}})} e^{ip_1(x-y)} e^{ip_2(y-z)}, \end{aligned} \quad (86)$$

where

$$Z_M^{\text{3pt}} = 1 + \frac{N_f}{M_{12}^2 F^2 V} + \frac{\langle \mathcal{G}(U_0) + \mathcal{H}(U_0) \rangle_{U_0}}{\langle \mathcal{E}(U_0) \rangle_{U_0}}, \quad (87)$$

$$\begin{aligned} \mathcal{G}(U_0) \equiv & \frac{1}{4} \left[\left(([U_0]_{22} + [U_0^\dagger]_{22} - 2)(2 + [U_0]_{11}[U_0]_{22} + [U_0^\dagger]_{11}[U_0^\dagger]_{22}) \right. \right. \\ & + 8([U_0]_{22} + [U_0^\dagger]_{22}) \\ & - 6[U_0]_{11}[U_0]_{22} - 6[U_0^\dagger]_{11}[U_0^\dagger]_{22} - 4[U_0]_{22}[U_0^\dagger]_{22} \\ & \left. \left. - ([U_0]_{22} + [U_0^\dagger]_{22} - 4)([U_0]_{12}[U_0^\dagger]_{21} + [U_0^\dagger]_{12}[U_0]_{21}) \right) \right] \end{aligned}$$

$$\begin{aligned}
& -([U_0]_{12}[U_0]_{21}[U_0]_{22} + [U_0^\dagger]_{12}[U_0^\dagger]_{21}[U_0^\dagger]_{22}) \\
& + 2([U_0]_{11} + [U_0^\dagger]_{11} - 2)(1 + [U_0]_{22}[U_0^\dagger]_{22}) \Big\} \\
& + ([U_0]_{11} + [U_0^\dagger]_{22})([U_0 \mathcal{M}^\dagger U_0]_{22}/m - 1) \\
& + ([U_0^\dagger]_{11} + [U_0]_{22})([U_0^\dagger \mathcal{M} U_0^\dagger]_{22}/m - 1) \\
& + 2[U_0]_{22}([U_0 \mathcal{M}^\dagger U_0]_{11}/m - 1) + 2[U_0^\dagger]_{22}([U_0^\dagger \mathcal{M} U_0^\dagger]_{11}/m - 1) \\
& - ([U_0]_{12}[U_0^\dagger]_{21}[U_0^\dagger]_{22} + [U_0^\dagger]_{12}[U_0]_{21}[U_0]_{22}) \\
& - ([U_0^\dagger]_{21}[U_0 \mathcal{M}^\dagger U_0]_{12}/m + [U_0]_{21}[U_0^\dagger \mathcal{M} U_0^\dagger]_{12}/m) \Big], \tag{88}
\end{aligned}$$

$$\mathcal{H}(U_0) \equiv -\frac{1}{2N_f} \Big[([U_0]_{12} + [U_0^\dagger]_{12})([U_0 \mathcal{M}^\dagger U_0]_{21}/m + [U_0^\dagger \mathcal{M} U_0^\dagger]_{21}/m) \Big]. \tag{89}$$

For the correction in the operators, we have a contribution from the L_9 term:

$$\begin{aligned}
& \langle \langle [[P(x)]_{12} [V_\mu(y)]_{ii} [P(x)]_{21}]^{L_9} \rangle_\xi \rangle_{U_0} = \\
& (\delta_{i2} - \delta_{i1}) \frac{i \Sigma_{\text{eff}}^2}{4 F_{\text{eff}}^2 V^2} \langle \mathcal{E}(U_0) \rangle_{U_0} \left(-\frac{2L_9}{F_{\text{eff}}^2} \right) \\
& \times \sum_{p_1 \neq 0} \sum_{p_2 \neq 0} \frac{i [p_2 \cdot (p_1 - p_2)] (p_1)_\mu - i [p_1 \cdot (p_1 - p_2)] (p_2)_\mu}{p_1^2 p_2^2} e^{ip_1(x-y)} e^{ip_2(y-z)}, \tag{90}
\end{aligned}$$

The correction from $\mathcal{S}_K^{\text{NLO}}$ term is obtained as

$$\begin{aligned}
& - \langle \langle [[P(x)]_{12} [V_\mu(y)]_{ii} [P(z)]_{21}]^{\text{LO}} \mathcal{S}_K^{\text{NLO}} \rangle_\xi \rangle_{U_0} = \\
& (\delta_{i2} - \delta_{i1}) \frac{i \Sigma_{\text{eff}}^2}{4 F_{\text{eff}}^2 V^2} \langle \mathcal{E}(U_0) \rangle_{U_0} \\
& \times \left(-\frac{N_f}{2 F_{\text{eff}}^2} \right) \sum_{p_1 \neq 0} \sum_{p_2 \neq 0} \frac{-i(p_1 + p_2)^\nu I_{\mu\nu}(-p_1^0 + p_2^0, -\mathbf{p}_1 + \mathbf{p}_2)}{p_1^2 p_2^2} e^{ip_1(x-y)} e^{ip_2(y-z)}, \tag{91}
\end{aligned}$$

where

$$I_{\mu\nu}(q_0, \mathbf{q}) \equiv \frac{1}{V} \sum_{p \neq 0, q} \frac{p^\mu (q^\nu - 2p^\nu)}{p^2 (q - p)^2} \quad (q^2 = q_0^2 + \mathbf{q}^2). \tag{92}$$

Now let us summarize all the above results for the $\mu = 0$ case, inserting momenta \mathbf{p}_f and \mathbf{p}_i . Using the notations $t = x_0 - y_0$, $t' = y_0 - z_0$, $E^{3\text{pt}}(\mathbf{p}) \equiv \sqrt{M_{12}^2 Z_M^{3\text{pt}} + \mathbf{p}^2}$, and

$$c(\mathbf{p}, t) = \frac{\cosh[E^{3\text{pt}}(\mathbf{p})(t - T/2)]}{2E^{3\text{pt}}(\mathbf{p}) \sinh[E^{3\text{pt}}(\mathbf{p})T/2]}, \quad s(\mathbf{p}, t) = \frac{\sinh[E^{3\text{pt}}(\mathbf{p})(t - T/2)]}{2E^{3\text{pt}}(\mathbf{p}) \sinh[E^{3\text{pt}}(\mathbf{p})T/2]}, \tag{93}$$

one can express the result as

$$\begin{aligned}
C^{PV_0P}(t, t'; \mathbf{p}_f, \mathbf{p}_i) &\equiv \langle [P(x_0, -\mathbf{p}_f)]_{12} V_0^3(y_0, \mathbf{q}) [P(z_0, \mathbf{p}_i)]_{21} \rangle \\
&= -\frac{L^3 \Sigma_{\text{eff}}^2}{4F_{\text{eff}}^2} \langle \mathcal{E}(U_0) \rangle_{U_0} \delta_{\mathbf{q}, \mathbf{p}_f - \mathbf{p}_i}^{(3)} Z_k F_V(q_0, \mathbf{q}) \\
&\quad \times \left[iE^{3\text{pt}}(\mathbf{p}_i) c(\mathbf{p}_f, t) s(\mathbf{p}_i, t') + iE^{3\text{pt}}(\mathbf{p}_f) s(\mathbf{p}_f, t) c(\mathbf{p}_i, t') \right]. \quad (94)
\end{aligned}$$

Here, as mentioned in the above calculations, we have omitted the two-pion-like propagations, and the $x_0 - z_0 = t + t'$ dependent long-distance correlators, as they are expected to be exponentially small.

The vector form factor $F_V(q_0, \mathbf{q})$ is given by

$$F_V(q_0, \mathbf{q}) = \frac{Z^{V2}}{(Z^\xi)^2} - \frac{2L_9}{F_{\text{eff}}^2} q^2 - \frac{N_f}{2F_{\text{eff}}^2} (l(q_0, \mathbf{q}) - l_{00}), \quad (95)$$

where $l(q_0, \mathbf{q})$ is a part of $I_{0\nu}(q_0, \mathbf{q})$ which is proportional to $\delta_{0\nu}$. Another part proportional to $q_0 q_\nu$ cannot contribute since it is contracted with a perpendicular vector \bar{q}^ν to q_μ . Namely, $l(q_0, \mathbf{q})$ is given by

$$l(q_0, \mathbf{q}) = I_{0\nu}(q_0, \mathbf{q}) \bar{q}^\nu / \bar{q}_0. \quad (96)$$

More details are discussed in Appendix B.

Note in the above formula, the (finite) renormalization factor

$$Z_k = 1 - \frac{N_f}{2F_{\text{eff}}^2} l_{00}, \quad l_{00} \equiv -\frac{1}{4\pi^2} \sum_{b \neq 0} \frac{1}{|b_\mu|^2} \left(1 - \frac{2(b_0)^2}{|b_\mu|^2} \right), \quad (97)$$

where the summation is taken over the vector $b_\mu = (n_0 T, n_1 L, n_2 L, n_3 L)$ with integers n_μ , is introduced so that $F_V(0, \mathbf{0}) = 1$ is maintained even in a finite volume. Therefore, the finite volume effects contained in $F_V(q_0, \mathbf{q})$ are only those which come from the non-zero modes, vanish in the $q_\mu \rightarrow 0$ limit, and are thus expected to be perturbatively small. We will discuss the details of the remaining finite volume effects in the next section.

Finally, let us discuss the renormalization of the above formula Eq. (95). Since the finite volume effects are free from UV divergences, it is sufficient to consider the $V \rightarrow \infty$ limit of $F_V(q_0, \mathbf{q})$. It is not difficult to see that the quadratic divergence in $Z^{V2}/(Z^\xi)^2$ is precisely canceled by that in $l(q_0, \mathbf{q})$. Therefore, we only need to renormalize the logarithmic divergence of $l(q_0, \mathbf{q})$ by the re-definition of L_9 .

Employing the dimensional regularization, we can easily evaluate its logarithmic divergence as

$$\begin{aligned}\lim_{V \rightarrow \infty} l(q_0, \mathbf{q}) &= \frac{\bar{q}^\nu}{\bar{q}_0} \int \frac{d^d p}{(2\pi)^d} \frac{-2p_0 p_\nu}{p^2(p-q)^2} \\ &= \frac{1}{16\pi^2} \left\{ \frac{q^2}{6} \left(\frac{2}{\epsilon} + 1 - \gamma_E + \ln 4\pi - \ln \mu_{sub}^2 \right) - \frac{q^2}{6} \ln \frac{q^2}{\mu_{sub}^2} + \frac{5}{18} q^2 \right\},\end{aligned}\quad (98)$$

where $\epsilon = 4 - d$, $\gamma_E = 0.57721 \dots$ is the Euler's constant, and μ_{sub} denotes the subtraction scale. This divergence can be absorbed in the renormalization of L_9 :

$$L_9^r(\mu_{sub}) \equiv L_9 - \frac{N_f}{12} \times \frac{1}{16\pi^2} \left(-\frac{1}{\epsilon} - \frac{1}{2}(-\gamma_E + \ln 4\pi + 1 - \ln \mu_{sub}^2) \right), \quad (99)$$

and one obtains the infinite volume limit for the vector form factor,

$$F_V^\infty(q_0, \mathbf{q}) = 1 - \frac{2L_9^r(\mu_{sub})}{F_{\text{eff}}^2} q^2 - \frac{N_f}{2F_{\text{eff}}^2} \frac{1}{16\pi^2} \left[-\frac{1}{6} q^2 \ln \frac{q^2}{\mu_{sub}^2} + \frac{5}{18} q^2 \right], \quad (100)$$

which agrees with the known (massless limit of) result within ChPT. Note that we cannot expect $F_V^\infty(q_0, \mathbf{q})$ to describe the lattice data well, since the physics beyond ChPT is omitted in the ChPT expression. However, we can still expect that the finite volume correction : $F_V(q_0, \mathbf{q}) - F_V^\infty(q_0, \mathbf{q})$ is well described within ChPT, which will be discussed in the next section.

V. EXTRACTION OF THE VECTOR FORM FACTOR OF PION

In this section, we show how to eliminate the leading zero-momentum pion mode's contribution from the correlator, and how to extract the vector form factor of pions. There still remain finite volume effects from non-zero modes but they are sub-leading contributions. From the one-loop calculation of the non-zero momentum modes, we numerically estimate this remaining effect, and show they are actually a small perturbation.

A. Removing dominant finite volume effects from the pion zero mode

In the previous section, we have neglected the t -independent or t' -independent terms in our calculation. In the final form Eq. (94), if both of \mathbf{p}_i and \mathbf{p}_f are non-zero, these terms are automatically dropped. However, if these momenta are zero, we have to take subtraction of the correlators at different time-slices, $\Delta_t f(t) \equiv f(t) - f(t_{\text{ref}})$, $\Delta_{t'} f(t') \equiv f(t') - f(t'_{\text{ref}})$, with

t_{ref} and t'_{ref} , respectively. Similar procedure was already shown in the two-point correlators. To keep $t_{\text{ref}} + t'_{\text{ref}} < T/2$ and $t, t' < t_{\text{ref}}$, which are the conditions to suppress the contribution from pions wrapping around the periodic space-time, $t_{\text{ref}} = t'_{\text{ref}} \sim T/4$ would be optimal. In the following, we take $t'_{\text{ref}} = t_{\text{ref}}$, for simplicity.

With the above time-slice subtraction in mind, and noting $F_V(0, \mathbf{0}) = 1$ the following ratios are useful for extracting the vector pion form factor:

$$\begin{aligned}
R_1(t, t'; \mathbf{p}_f, \mathbf{p}_i) &\equiv \frac{C^{PV_0P}(t, t'; \mathbf{p}_f, \mathbf{p}_i)}{\Delta_t \Delta_{t'} C^{PV_0P}(t, t'; \mathbf{0}, \mathbf{0})} \\
&= F_V(q_0, \mathbf{q}) \times \frac{E^{3\text{pt}}(\mathbf{p}_i) c(\mathbf{p}_f, t) s(\mathbf{p}_i, t') + E^{3\text{pt}}(\mathbf{p}_f) s(\mathbf{p}_f, t) c(\mathbf{p}_i, t')}{E^{3\text{pt}}(\mathbf{0}) \Delta_t c(\mathbf{0}, t) \Delta_{t'} s(\mathbf{0}, t') + E^{3\text{pt}}(\mathbf{0}) \Delta_t s(\mathbf{0}, t) \Delta_{t'} c(\mathbf{0}, t')}, \\
R_2(t, t'; \mathbf{0}, \mathbf{p}_i) &\equiv \frac{\Delta_t C^{PV_0P}(t, t'; \mathbf{0}, \mathbf{p}_i)}{\Delta_t \Delta_{t'} C^{PV_0P}(t, t'; \mathbf{0}, \mathbf{0})} \\
&= F_V(q_0, \mathbf{q}) \times \frac{E^{3\text{pt}}(\mathbf{p}_i) \Delta_t c(\mathbf{0}, t) s(\mathbf{p}_i, t') + E^{3\text{pt}}(\mathbf{0}) \Delta_t s(\mathbf{0}, t) c(\mathbf{p}_i, t')}{E^{3\text{pt}}(\mathbf{0}) \Delta_t c(\mathbf{0}, t) \Delta_{t'} s(\mathbf{0}, t') + E^{3\text{pt}}(\mathbf{0}) \Delta_t s(\mathbf{0}, t) \Delta_{t'} c(\mathbf{0}, t')}.
\end{aligned} \tag{101}$$

Note here that the t and t' dependences are uniquely determined once $M_{12} \sqrt{Z_M^{3\text{pt}}}$ is given. Therefore, $F_V(q_0, \mathbf{q})$ can be extracted by performing a one-parameter fit at a long-distance, taking $M_{12} \sqrt{Z_M^{3\text{pt}}}$ as a free parameter.

In the numerical lattice analysis, one could also try taking further ratios with two-point functions. Namely,

$$\begin{aligned}
R'_1(t, t'; \mathbf{p}_f, \mathbf{p}_i) &\equiv \frac{C^{PV_0P}(t, t'; \mathbf{p}_f, \mathbf{p}_i)}{\Delta_t \Delta_{t'} C^{PV_0P}(t, t'; \mathbf{0}, \mathbf{0})} \\
&\times \left(\frac{-\Delta_t C_{PP}^{2\text{pt}}(t, \mathbf{0}) \Delta_{t'} \partial_{t'} C_{PP}^{2\text{pt}}(t', \mathbf{0}) - \Delta_t \partial_t C_{PP}^{2\text{pt}}(t, \mathbf{0}) \Delta_{t'} C_{PP}^{2\text{pt}}(t', \mathbf{0})}{(E^{2\text{pt}}(\mathbf{p}_i) + E^{2\text{pt}}(\mathbf{p}_f)) C_{PP}^{2\text{pt}}(t, \mathbf{p}_i) C_{PP}^{2\text{pt}}(t', \mathbf{p}_f)} \right), \\
R'_2(t, t'; \mathbf{0}, \mathbf{p}_i) &\equiv \frac{\Delta_t C^{PV_0P}(t, t'; \mathbf{0}, \mathbf{p}_i)}{\Delta_t \Delta_{t'} C^{PV_0P}(t, t'; \mathbf{0}, \mathbf{0})} \\
&\times \left(\frac{-\Delta_t C_{PP}^{2\text{pt}}(t, \mathbf{0}) \Delta_{t'} \partial_{t'} C_{PP}^{2\text{pt}}(t', \mathbf{0}) - \Delta_t \partial_t C_{PP}^{2\text{pt}}(t, \mathbf{0}) \Delta_{t'} C_{PP}^{2\text{pt}}(t', \mathbf{0})}{C_{PP}^{2\text{pt}}(t', \mathbf{p}_i) [-\Delta_t \partial_t C_{PP}^{2\text{pt}}(t, \mathbf{0}) + E(\mathbf{p}_i) \Delta_t C_{PP}^{2\text{pt}}(t, \mathbf{0})]} \right).
\end{aligned} \tag{102}$$

Note that $E^{2\text{pt}}(\mathbf{p}) = E^{3\text{pt}}(\mathbf{p})$ at LO. At NLO, their expressions are different, reflecting the different zero-mode integrals. However, they are numerically very similar to each other with reasonable set-ups of the lattice simulation parameters. In particular, they share the exactly same chiral limit, and the infinite volume limit as seen in Figure 1. Therefore, these ratios $R'_1(t, t'; \mathbf{p}_f, \mathbf{p}_i)$, and $R'_2(t, t'; \mathbf{0}, \mathbf{p}_i)$ should cancel the t and t' dependences, and directly give the values of $F_V(q_0, \mathbf{q})$.

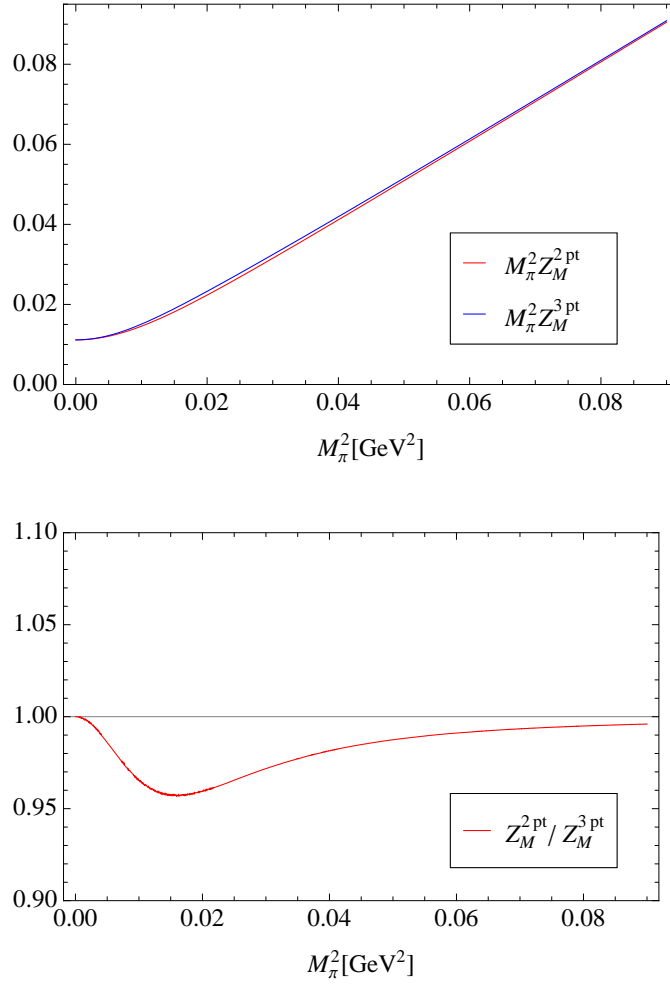


FIG. 1: Numerical estimates for the *pion mass squared* $M_\pi^2 Z_M^{2pt}$ and $M_\pi^2 Z_M^{3pt}$ (top panel) and their ratio Z_M^{2pt}/Z_M^{3pt} (bottom). Here, we use $L = T/2=2\text{fm}$, and $F_{\text{eff}} = 92.2 \text{ MeV}$ as inputs.

JLQCD collaboration [6] has employed the latter ratios and found a good plateau for it, extracting a pion charge radius, which is consistent with the experiment.

It should be noted that except for Z_M^{3pt} , which is essentially irrelevant in both of the above ratios, we don't need any zero-mode integrals which could have been a complicated combination of Bessel functions. The remaining finite volume effect in $F_V(q_0, \mathbf{q})$ is a perturbative correction from the non-zero modes only and thus, is expected to be small as shown in the next subsection.

B. Remaining Finite Volume Effects from non-zero modes

After removing the dominant finite volume effect from the zero-mode, what remains in $F_V(q_0, \mathbf{q})$ is the effect of the non-zero momentum modes, which is expected to be perturbatively small. In this subsection, we compute this non-zero-momentum effect to the pion 1-loop and numerically confirm this expectation.

To this end, all we need to evaluate is

$$I_{\mu\nu}(q_0, \mathbf{q}) = \frac{1}{V} \sum_{p \neq 0, q} \frac{-2p_\mu p_\nu}{p^2(p-q)^2}. \quad (103)$$

Here and in the following, we ignore the terms proportional to q_ν , since they are always contracted with a perpendicular 4-momentum vector to q_μ , and thus do not contribute to the final result.

It is not difficult to decompose it as

$$I_{\mu\nu}(q_0, \mathbf{q}) = \sum_{b_\mu = n_\mu L_\mu} I_{\mu\nu}^b(q_0, \mathbf{q}), \quad (104)$$

where

$$I_{\mu\nu}^b(q_0, \mathbf{q}) \equiv \int \frac{d^4 p}{(2\pi)^4} e^{ipb} \frac{-2p_\mu p_\nu}{p^2(p-q)^2}. \quad (105)$$

Note that $I^{b=0}(q_0, \mathbf{q})$ is the infinite volume limit of $I_{\mu\nu}(q_0, \mathbf{q})$ and thus, the finite volume correction is given by

$$\Delta I_{\mu\nu}(q_0, \mathbf{q}) = \sum_{b \neq 0} I_{\mu\nu}^b(q_0, \mathbf{q}). \quad (106)$$

In the standard manner, each contribution $I_{\mu\nu}^b(q_0, \mathbf{q})$ can be computed as

$$\begin{aligned} I_{\mu\nu}^b(q_0, \mathbf{q}) &= 2 \frac{\partial}{\partial b^\mu} \frac{\partial}{\partial b^\nu} \int_0^1 dx e^{ixbq} \int \frac{d^4 p}{(2\pi)^4} \frac{e^{ipb}}{(p^2 + \Delta)^2} \\ &= -\frac{1}{4\pi^2} \int_0^1 dx e^{ixbq} \left[\frac{\delta_{\mu\nu}}{|b_\mu|} \sqrt{\Delta} K_1(\sqrt{\Delta}|b_\mu|) - \frac{b_\mu b_\nu}{|b_\mu|^2} \Delta K_2(\sqrt{\Delta}|b_\mu|) \right], \end{aligned} \quad (107)$$

where $\Delta = x(1-x)q^2$, and $K_i(z)$ denotes the i -th modified Bessel function. Here, we have neglected a term proportional to $q_\mu b_\nu$, since that term is proportional to q_ν after the summation over b_ν .

When $b_0 = 0$, it is straight forward to numerically evaluate the above form. However, when $b_0 \neq 0$, we need a special care because we need to analytically continue the results with respect to q_0 . Here we simplify the situation using an inequality

$$\left| \int_0^1 dx e^{i\alpha} f(x) \right| < \left| \int_0^1 dx |e^{i\alpha}| f(x) \right| = \left| \int_0^1 dx f(x) \right|, \quad (108)$$

in Eq. (107). Namely we neglect the oscillating factor $\exp(ixb_0q_0)$. Then the analytic continuation of q_0 has no subtlety since the Bessel functions are all vanishing in the limit $|q_0| \rightarrow \infty$ with any complex phase. Note here that the real part $\sqrt{\Delta}$ is always positive. We do not think this *over-estimation* affects the result very much, since the temporal direction is usually larger than the spacial direction by a factor of 2 or 3, and therefore, the contribution from $b_0 \neq 0$ is much smaller from the beginning.

Taking $\mu = 0$ direction the finite volume correction to $F_V(q_0, \mathbf{q})$ can be computed as

$$\begin{aligned} \Delta F_V(q_0, \mathbf{q}) &\equiv F_V(q_0, \mathbf{q}) - F_V^\infty(q_0, \mathbf{q}) \\ &= -\frac{N_f}{2F_{\text{eff}}^2} (\Delta l(q_0, \mathbf{q}) - l_{00}), \end{aligned} \quad (109)$$

where

$$\Delta l(q_0, \mathbf{q}) = -\frac{1}{4\pi^2} \sum_{b_\mu} \int_0^1 dx e^{ix\mathbf{b} \cdot \mathbf{q}} \left[\frac{\sqrt{\Delta}}{|b_\mu|} K_1(\sqrt{\Delta}|b_\mu|) - \frac{b_0^2}{|b_\mu|^2} \Delta K_2(\sqrt{\Delta}|b_\mu|) \right]. \quad (110)$$

Note that $\Delta l(0, \mathbf{0}) = l_{00}$.

Our numerical estimates for $\Delta F_V(q_0, \mathbf{q})$ at $L = T/2 = 2, 3, 4$ fm are presented in Fig. 2. Here, we denote $q^0 = i \left(\sqrt{\mathbf{p}_f^2 + M_\pi^2} - \sqrt{\mathbf{p}_i^2 + M_\pi^2} \right)$, assuming the dispersion relation of the pion energy, $\mathbf{q} = \mathbf{p}_f - \mathbf{p}_i$, and choose $M_\pi = 135 \text{ MeV}$, $F_{\text{eff}} = 92.2 \text{ MeV}$, as inputs. The zig-zag behavior may be due to the lack of the rotational symmetry on the lattice. Since $F_V^\infty(q^2)$ is an $\mathcal{O}(1)$ quantity, our result shows the remaining finite volume effects is around a few % already at $L = 3$ fm, even when $m_\pi L < 1$.

VI. SUMMARY AND DISCUSSION

We have studied finite volume effects on the electro-magnetic pion form factor in the ϵ regime. The pseudoscalar-vector-pseudoscalar three point function has been calculated in the ϵ expansion of chiral perturbation theory to the next-to-leading order.

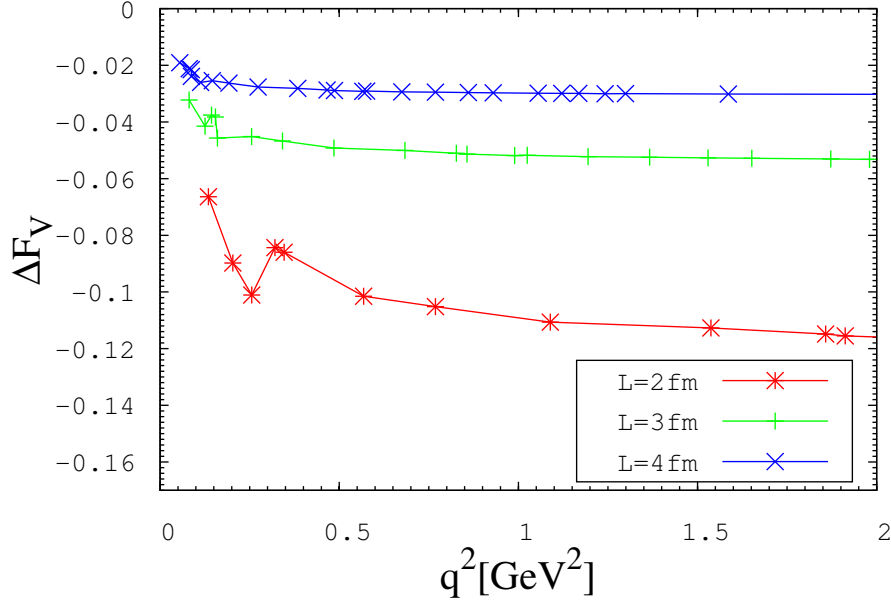


FIG. 2: Numerical estimates for ΔF_V .

The dominant finite volume effects, which come from the zero-mode of the pions can be removed by two simple manipulations: by inserting non-zero momentum to relevant operators (or making a subtraction at different time correlators) and taking an appropriate ratio of them. After these manipulations, one can safely extract the electro-magnetic pion form factor for which the remaining finite volume correction from the non-zero modes is suppressed to a few percent level already at $L = 3$ fm even in the ϵ regime (see Figure 2).

It is important to note that our analysis has been done without using any special features of the ϵ expansion, and the dominance of the zero-mode contribution is expected to be a common feature of finite volume effects in any regime of QCD. Therefore, our method can be useful for simulations in the p regime, including the ones with twisted boundary conditions [28, 29]. We also expect a wide application to other quantities like form factors of heavier hadrons.

The authors thank P. H. Damgaard, S. Hashimoto, T. Onogi, S. Yamaguchi for useful discussions. The work of HF is supported by the Grant-in-Aid of the Japanese Ministry of Education (No. 25800147).

[1] J. Gasser and H. Leutwyler, *Annals Phys.* **158**, 142 (1984);

- [2] J. Gasser and H. Leutwyler, Nucl. Phys. B **250**, 465 (1985).
- [3] B. B. Brandt, Int. J. Mod. Phys. E **22**, 1330030 (2013) [arXiv:1310.6389 [hep-lat]].
- [4] J. Koponen *et al.*, arXiv:1311.3513 [hep-lat].
- [5] H. Fukaya *et al.*, (JLQCD Collaboration), arXiv:1211.0743 [hep-lat].
- [6] H. Fukaya, S. Aoki, S. Hashimoto, T. Kaneko, H. Matsufuru and J. Noaki, Phys. Rev. D **90**, 034506 (2014) [arXiv:1405.4077 [hep-lat]].
- [7] M. Kieburg, J. J. M. Verbaarschot and S. Zafeiropoulos, Phys. Rev. D **88**, 094502 (2013) [arXiv:1307.7251 [hep-lat]].
- [8] S. Aoki *et al.*, arXiv:1310.8555 [hep-lat].
- [9] J. Gasser and H. Leutwyler, Phys. Lett. B **184**, 83 (1987).
- [10] H. Fukaya and T. Suzuki, arXiv:1402.2722 [hep-lat].
- [11] F. C. Hansen, Nucl. Phys. B **345**, 685 (1990).
- [12] F. C. Hansen and H. Leutwyler, Nucl. Phys. B **350**, 201 (1991).
- [13] P. Hernandez and M. Laine, JHEP **0301**, 063 (2003) [hep-lat/0212014].
- [14] L. Giusti, P. Hernandez, M. Laine, P. Weisz and H. Wittig, JHEP **0411**, 016 (2004) [hep-lat/0407007].
- [15] F. Bernardoni, P. H. Damgaard, H. Fukaya and P. Hernandez, JHEP **0810**, 008 (2008) [arXiv:0808.1986 [hep-lat]].
- [16] P. Hernandez, M. Laine, C. Pena, E. Torro, J. Wennekers and H. Wittig, JHEP **0805**, 043 (2008) [arXiv:0802.3591 [hep-lat]].
- [17] S. Aoki and H. Fukaya, Phys. Rev. D **84**, 014501 (2011) [arXiv:1105.1606 [hep-lat]].
- [18] H. Leutwyler and A. V. Smilga, Phys. Rev. D **46**, 5607 (1992).
- [19] K. Splittorff and J. J. M. Verbaarschot, Phys. Rev. Lett. **90**, 041601 (2003).
- [20] Y. V. Fyodorov and G. Akemann, JETP Lett. **77**, 438 (2003).
- [21] S. Aoki *et al.* [JLQCD and TWQCD Collaboration], Phys. Rev. D **80**, 034508 (2009).
- [22] T. Kaneko *et al.* [JLQCD Collaboration], PoS **LATTICE2010**, 146 (2010).
- [23] F. Bernardoni and P. Hernandez, JHEP **0710**, 033 (2007) [arXiv:0707.3887 [hep-lat]].
- [24] S. Aoki and H. Fukaya, Phys. Rev. D **81**, 034022 (2010) [arXiv:0906.4852 [hep-lat]].
- [25] C. Bernard [MILC Collaboration], Phys. Rev. D **65**, 054031 (2002) [arXiv:hep-lat/0111051].
- [26] P. Hasenfratz and H. Leutwyler, Nucl. Phys. B **343**, 241 (1990).
- [27] P. H. Damgaard and H. Fukaya, Nucl. Phys. B **793**, 160 (2008) [arXiv:0707.3740 [hep-lat]].

[28] T. Mehen and B. C. Tiburzi, Phys. Rev. D **72**, 014501 (2005) [hep-lat/0505014].

[29] B. C. Tiburzi, arXiv:1407.4059 [hep-lat].

Appendix A: Zero-mode integral

In this appendix, we evaluate the U_0 integrals which are necessary for numerical estimation of $Z_M^{2\text{pt}}$ or $Z_M^{3\text{pt}}$. Although our analysis in this paper is done only in the unquenched QCD, we use the partially quenched results by [19, 20], because some expressions are simpler for the partially quenched results, and the results would be easily extended to the partially quenched study in these expressions. The unquenched results are obtained simply by setting the valence quark mass m_v to the one of the sea quark masses.

We start with the so-called graded partition function which consists of n bosons and m fermions. Its non-perturbative analytic form is given by [19, 20]

$$\mathcal{Z}_{n,m}^Q(\{\mu_i\}) = \frac{\det[\mu_i^{j-1} \mathcal{J}_{Q+j-1}(\mu_i)]_{i,j=1,\dots,n+m}}{\prod_{j>i=1}^n (\mu_j^2 - \mu_i^2) \prod_{j>i=n+1}^{n+m} (\mu_j^2 - \mu_i^2)}, \quad (\text{A1})$$

in a fixed topological sector of Q . Here \mathcal{J} 's are defined as $\mathcal{J}_{Q+j-1}(\mu_i) \equiv (-1)^{j-1} K_{Q+j-1}(\mu_i)$ for $i = 1, \dots, n$ and $\mathcal{J}_{Q+j-1}(\mu_i) \equiv I_{Q+j-1}(\mu_i)$ for $i = n+1, \dots, n+m$, where K_ν and I_ν are the modified Bessel functions. Partial quenching is completed by taking the boson masses to those of valence fermions.

Integrals of some diagonal matrix elements are obtained by simply differentiating the partition function,

$$\begin{aligned} \mathcal{S}_v &\equiv \frac{1}{2} \left\langle [U_0]_{vv} + [U_0^\dagger]_{vv} \right\rangle_{U_0} = \lim_{\mu_b \rightarrow \mu_v} \frac{\partial}{\partial \mu_v} \ln \mathcal{Z}_{1,1+N_f}^Q(\mu_b, \mu_v, \{\mu_{sea}\}), \\ \mathcal{D}_v &\equiv \frac{1}{4} \left\langle \left([U_0]_{vv} + [U_0^\dagger]_{vv} \right)^2 \right\rangle_{U_0} \\ &= \frac{1}{\mathcal{Z}_{N_f}^Q(\{\mu_{sea}\})} \lim_{\mu_b \rightarrow \mu_v} \frac{\partial^2}{\partial \mu_v^2} \mathcal{Z}_{1,1+N_f}^Q(\mu_b, \mu_v, \{\mu_{sea}\}), \\ \mathcal{D}_{v_1 v_2} &\equiv \frac{1}{4} \left\langle \left([U_0]_{v_1 v_1} + [U_0^\dagger]_{v_1 v_1} \right) \left([U_0]_{v_2 v_2} + [U_0^\dagger]_{v_2 v_2} \right) \right\rangle_{U_0} \\ &= \frac{1}{\mathcal{Z}_{N_f}^Q(\{\mu_{sea}\})} \lim_{\mu_{b_1} \rightarrow \mu_{v_1}, \mu_{b_2} \rightarrow \mu_{v_2}} \frac{\partial}{\partial \mu_{v_1}} \frac{\partial}{\partial \mu_{v_2}} \mathcal{Z}_{2,2+N_f}^Q(\mu_{b_1}, \mu_{b_2}, \mu_{v_1}, \mu_{v_2}, \{\mu_{sea}\}), \quad (\text{A2}) \end{aligned}$$

and

$$\mathcal{T}_{v_1 v_2} \equiv \frac{1}{8} \left\langle \left([U_0]_{v_1 v_1} + [U_0^\dagger]_{v_1 v_1} \right)^2 \left([U_0]_{v_2 v_2} + [U_0^\dagger]_{v_2 v_2} \right) \right\rangle_{U_0}$$

$$= \frac{1}{\mathcal{Z}_{N_f}^Q(\{\mu_{sea}\})} \lim_{\mu_{b_1} \rightarrow \mu_{v_1}, \mu_{b_2} \rightarrow \mu_{v_2}} \frac{\partial^2}{\partial \mu_{v_1}} \frac{\partial}{\partial \mu_{v_2}} \mathcal{Z}_{2,2+N_f}^Q(\mu_{b_1}, \mu_{b_2}, \mu_{v_1}, \mu_{v_2}, \{\mu_{sea}\}) . \quad (\text{A3})$$

Then, U_0 integrals for the degenerate case $m_1 = m_2$ can be written as

$$\langle \mathcal{B}(U_0) \rangle = 2 \left[1 + \frac{Q^2}{\mu_1} - \frac{2}{N_f} \mathcal{D}_{11} + \left(1 + \frac{2}{N_f} \right) \mathcal{D}_{11} \right], \quad (\text{A4})$$

$$\langle \mathcal{D}^0(U_0) \rangle_{U_0} = 4\mathcal{S}_1, \quad (\text{A5})$$

$$\langle \mathcal{D}^1(U_0) \rangle_{U_0} = \frac{N_f}{\mu_1} \left(1 - \mathcal{D}_{11} - \frac{Q^2}{\mu_1^2} \right), \quad (\text{A6})$$

$$\langle \mathcal{D}^2(U_0) \rangle_{U_0} = -\frac{2}{\mu_1} \left(\partial_1 \mathcal{S}_1 - \frac{\mathcal{S}_1}{\mu_1} - \frac{2Q^2}{\mu_1} \mathcal{S}_1 \right), \quad (\text{A7})$$

$$\langle \mathcal{D}^3(U_0) \rangle_{U_0} = -\frac{4}{\mu_1} \left(\frac{1}{N_f} - \frac{\mathcal{D}_{11}}{N_f} - \frac{3Q^2}{N_f \mu_1^2} - \partial_1 \mathcal{S}_1 \right), \quad (\text{A8})$$

$$\langle \mathcal{D}^4(U_0) \rangle_{U_0} = \frac{4}{N_f^2} \partial_1 \mathcal{D}_1 + \frac{2}{\mu_1} \partial_1 \mathcal{S}_1 + \frac{4(N_f - 2)}{N_f^2} \partial_1 \mathcal{D}_{1j}|_{m_j=m_1},$$

$$\langle \mathcal{E}(U_0) \rangle_{U_0} = 2 \left(1 + 3\mathcal{D}_{11} + \frac{Q^2}{\mu_1^2} \right), \quad (\text{A9})$$

$$\begin{aligned} \langle \mathcal{G}(U_0) \rangle_{U_0} = 2 \left[\mathcal{T}_{11} - \frac{\partial_1 \mathcal{D}_1}{2} - \frac{3\mathcal{D}_1}{2\mu_1} + \left(-3 + \frac{-4N_f + 3}{2\mu_1} \right) \mathcal{D}_{11} \right. \\ \left. + \left(3 + \frac{3}{2\mu_1^2} + \frac{3Q^2}{\mu_1^2} \right) \mathcal{S}_1 - 1 - \frac{Q^2}{\mu_1^2} \left(1 + \frac{N_f}{\mu_1} \right) \right], \end{aligned} \quad (\text{A10})$$

$$\langle \mathcal{H}(U_0) \rangle_{U_0} = -4 \left[\frac{1 - \mathcal{D}_{11}}{2N_f \mu_1} - \frac{\partial_1 \mathcal{S}_1}{\mu_1} - \frac{3Q^2}{2\mu_1^3 N_f} \right], \quad (\text{A11})$$

Here, we have used

$$\lim_{\mu_1 \rightarrow \mu_2} \frac{\mathcal{S}_1 - \mathcal{S}_2}{\mu_1 - \mu_2} = \partial_1 \mathcal{S}_1 \quad (\text{A12})$$

Note that the derivative ∂_v is taken w.r.t the valence degree of freedom *after* $\mu_b = \mu_v$ limit is taken. This partially quenched expression is simpler than that of unquenched theory, as shown in Ref. [27].

It is also useful to note

$$\mathcal{D}_{11} \equiv \lim_{\mu_2 \rightarrow \mu_1} \mathcal{D}_{12} = - \frac{1}{\mathcal{Z}_{0,N_f}^Q(\mu_{sea})} \frac{\partial}{\partial \mu_b} \frac{\partial}{\partial \mu_v} \mathcal{Z}_{1,1+N_f}^Q(\mu_b, \mu_v, \{\mu_{sea}\}) \Bigg|_{\mu_b=\mu_v=\mu_1}, \quad (\text{A13})$$

which was shown in Appendix of Ref [17]. With this, the following non-trivial relations are obtained,

$$\partial_1 \mathcal{S}_1 = \mathcal{D}_1 - \mathcal{D}_{11}, \quad \partial_1^2 \mathcal{S}_1 = \partial_1 \mathcal{D}_1 - 2\partial_1 \mathcal{D}_{12}|_{m_2=m_1}. \quad (\text{A14})$$

Similarly, we can use

$$\mathcal{T}_{11} \equiv \lim_{\mu_2 \rightarrow \mu_1} \mathcal{T}_{21} = - \frac{1}{\mathcal{Z}_{0,N_f}^Q(\mu_{sea})} \frac{\partial}{\partial \mu_b} \frac{\partial^2}{\partial^2 \mu_v} \mathcal{Z}_{1,1+N_f}^Q(\mu_b, \mu_v, \{\mu_{sea}\}) \Big|_{\mu_b=\mu_v=\mu_1}. \quad (\text{A15})$$

Appendix B: Loop momentum summations

In the calculation of the one-loop diagram, we have encountered the momentum summation:

$$I_{\mu\nu}(q_0, \mathbf{q}) = \frac{1}{V} \sum_{p \neq 0, q} \frac{p^\mu (q^\nu - 2p^\nu)}{p^2 (q - p)^2} \quad (q^2 = q_0^2 + \mathbf{q}^2). \quad (\text{B1})$$

From the symmetry, on a finite volume $V = TL^3$ we can decompose it as

$$I_{\mu\nu}(q_0, \mathbf{q}) = \delta_{\mu\nu} I_1(q_0, \mathbf{q}) + \delta_{\mu 0} \delta_{\nu 0} I_2(q_0, \mathbf{q}) + q_\mu q_\nu I_3(q_0, \mathbf{q}). \quad (\text{B2})$$

Note that another possible choice $\sum_{i=1}^3 \delta_{\mu i} \delta_{\nu i}$ is not independent from the others since $\delta_{\mu\nu} = \delta_{\mu 0} \delta_{\nu 0} + \sum_{i=1}^3 \delta_{\mu i} \delta_{\nu i}$.

For a vector \bar{q}_μ which satisfy $q \cdot \bar{q} = 0$, we can simplify

$$I_{\mu\nu}(q_0, \mathbf{q}) \bar{q}^\nu = \bar{q}_\mu I_1(q_0, \mathbf{q}) + \delta_{\mu 0} \bar{q}_0 I_2(q_0, \mathbf{q}). \quad (\text{B3})$$

In particular, it is useful to note

$$I_{0\nu}(q_0, \mathbf{q}) \bar{q}^\nu = \bar{q}_0 l(q_0, \mathbf{q}), \quad (\text{B4})$$

where

$$l(q_0, \mathbf{q}) \equiv I_1(q_0, \mathbf{q}) + I_2(q_0, \mathbf{q}). \quad (\text{B5})$$